In conclusion it is deduced that the present forward-marching technique is quite accurate, provided the magnitude of the reverse flow velocity is less than 0.10 U_∞. In addition, a modification to the Thomas algorithm has been introduced which gives unconditional diagonal dominance and results in a column iterative scheme which has linear stability, at least in the present calculations.

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Bending of Cylindrically Anisotropic Sector Plates

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Introduction

It is the purpose of this Note to develop deflection and bending stress equations for a cylindrically anisotropic sector which is clamped along its radial edges and may have any boundary conditions along its circular edges. Results are desired for a variety of combinations of boundary conditions, including the case where the inner radius is clamped and the outer radius has a uniform shearing force along its edge. This is the case of a disk-type rotary regenerator matrix subject to a pressure drop and peripheral rubbing seal load.[1-3]

A thin plate of the form shown in Fig. 1 has an inner radius a and an outer radius b; the sector angle is θ. Coordinate r is measured from point O, and coordinate θ is measured from the center radial line of the sector with maximum and minimum values at $\theta = \pm\alpha/2$. The applied force is a uniformly distributed load acting normal to the surface of the plate and having a value of q psi. The boundary conditions are clamped on the circular edges and general along the radial lines.

The deflection is described in terms of two separate functions of the two variables, r and θ. One of the functions is assumed such that the boundary conditions will be satisfied, and the theorem of minimum potential energy[4,5] is used to obtain a differential equation in terms of the other variable alone. Solving this equation provides the two separate functions which, when combined, give an expression for the deflection.

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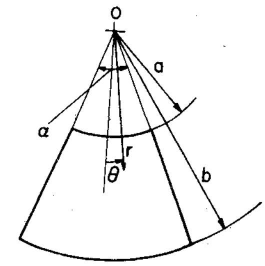
Fig. 1 Sector plate.



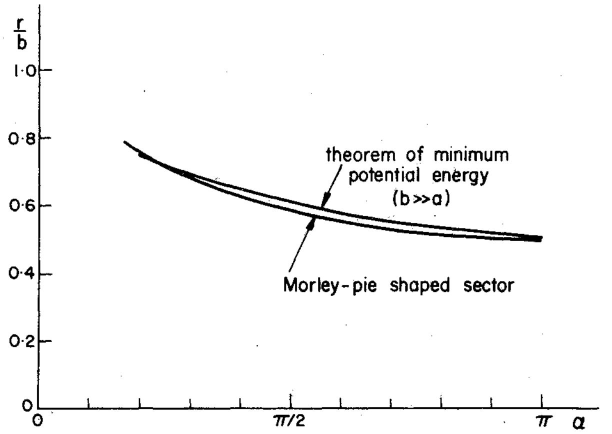


Fig. 2 Location of maximum plate deflection vs sector angle. (Completely clamped, uniformly loaded, isotropic sector, $\nu = 0.3$.)

Some preliminary discussion on the choice of functions is necessary here. The expression for the deflection is:

$$w(r,\theta) = f(r)g(\theta) \tag{1}$$

where $f(r)$ and $g(\theta)$ are functions of the single variables r and θ, respectively. The functions $f(r)$ and $g(\theta)$ must satisfy all the boundary conditions after being multiplied. In general, $f(r)$ must also be a function of the sector angle, α, since it would not be expected that the deflection curve along a line of constant θ would be similar for all sector angles. In fact, Morley[5] has shown that the location of the maximum deflection along the center radial line of an isotropic clamped pie-shaped sector will vary with the sector angle (see Fig. 2). Ben-Amoz assumes a function $f(r)$ for a clamped isotropic pie-shaped sector and includes what appears to be an empirical function of the sector angle in this function. This assumption gives good results until the sector angle becomes small. As the sector angle approaches zero, the maximum deflection approaches the outer radius. This clearly cannot occur for a clamped sector. In the case of the anisotropic plate, the problem becomes more complex since the function $f(r)$ should depend on the anisotropy of the plate as well as the sector angle.

To avoid the above problems, the authors have assumed a function for $g(\theta)$ instead of $f(r)$. There are several advantages to this. The deflections and stresses will be symmetric about the center radial line; thus a simple symmetric function of θ may be chosen without any concern for the sector angle or anisotropy. In addition, when the function $f(r)$ is obtained using the theorem of minimum potential energy, it will depend on all the system parameters that it should depend on, including the sector angle and the plate anisotropy. The results are compared to Morley's data for the case of the isotropic plate and good correlation is obtained.

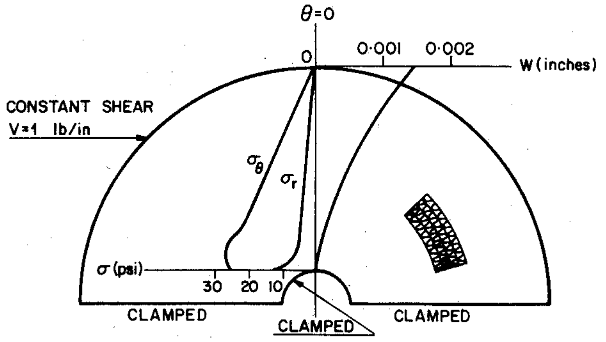


Fig. 3 Data for disk-type rotary regenerator matrix made of a Pyroceram glass ceramic structure (deflections and stress are shown for center radial line): $W_{max}=0.0014$ in.; $\sigma_{r_{max}}=13.09$ psi; and $\sigma_{\theta_{max}}=26.41$ psi.

The Problem

An expression for $g(\theta)$ is assumed such that it satisfies the clamped boundary conditions on the radial edges: $w=w'=0$ at $\theta=\pm\alpha/2$. The assumed deflection is then

$$w(r,\theta)=[(\theta^2/\alpha)-(\alpha/4)]^2 f(r) \tag{2}$$

The total potential energy of a thin orthotropic plate with a uniformly distributed load is

$$V=\int_{\theta=-\alpha/2}^{+\alpha/2}\int_{r=a}^{b}\{\tfrac{1}{2}[D_r(\frac{\partial^2 w}{\partial r^2})^2+D_\theta\frac{1}{r^2}(\frac{\partial w}{\partial r}+\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2})^2+D_r\nu_\theta\frac{2}{r}\frac{\partial^2 w}{\partial r^2}(\frac{\partial w}{\partial r}+\frac{1}{r}\frac{\partial^2 w}{\partial\theta^2})+D_k\frac{2}{r^2}(\frac{\partial^2 w}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial w}{\partial\theta})^2]-qw\}rdrd\theta \tag{3}$$

where $D_r=E_rh^3/12(1-\nu_r\nu_\theta)$, $D_\theta=E_\theta h^3/12(1-\nu_r\nu_\theta)$, and $D_k=G_{r\theta}h^3/12$.

When Eq. (2) is substituted into Eq. (3), an integration with respect to θ is possible. The resulting potential energy expression will be a function of r alone. According to the theorem of minimum potential energy, the variation of the potential energy, δV, equals zero for any δf. Using this principle and standard variational techniques, a fourth-order ordinary differential equation with constant coefficients is obtained

$$f^{IV}+\frac{2f'''}{r}+\phi\frac{f''}{r^2}-\phi\frac{f'}{r^3}+\lambda\frac{f}{r^4}=\beta \tag{4}$$

where

$$\phi=-\frac{24D_{r\theta}}{\alpha^2 D_r}-\frac{D_\theta}{D_r},\ \lambda=\frac{504D_\theta}{\alpha^4 D_r}-\frac{24}{\alpha^2 D_r}(D_{r\theta}+D_\theta),\ \text{and}\ \beta\ \frac{21q}{\alpha^2 D_r}$$

The particular solution of this equation is

$$f_p=\beta r^4/(72+8\phi+\lambda) \tag{5}$$

The homogeneous solution will depend upon the roots of the characteristic equation

$$n^4+(\phi-1)n^2-\phi+\lambda=0 \tag{6}$$

which is obtained by substituting $f=r^{n+1}$ into the homogeneous differential equation. The roots of the above quartic equation are:

$$n_{1,2,3,4}=\pm\sqrt{\frac{1-\phi}{2}\pm\sqrt{(\frac{1-\phi}{2})^2-(\lambda-\phi)}} \tag{7}$$

For sector angles $0<\alpha<\pi$ the four roots will be either 1) all real, 2) two pairs of complex conjugates $(\pm x\pm yi)$, or 3) two sets of real and equal roots $(\pm n,\pm n)$.

Thus, the function $f(r)$ can take one of three possible forms. The possible deflection functions are obtained from Eq. (2) (where the subscripts on w refer to the three possibilities listed):

$$w_1(r,\theta)=[\frac{\theta^2}{\alpha}-\frac{\alpha}{4}]^2 r[\frac{\beta r^3}{72+8\phi+\lambda}+\sum_{i=1}^{4}K_i r^{n_i}] \tag{8a}$$

$$w_2(r,\theta)=[\frac{\theta^2}{\alpha}-\frac{\alpha}{4}]^2 r\{\frac{\beta r^3}{72+8\phi+\lambda}+K_1[r^{x_1}\cos(y\ell nr)]+K_2[r^{x_1}\sin(y\ell nr)]+K_3[r^{-x_2}\cos(y\ell nr)]+K_4[r^{-x_2}\sin(y\ell nr)]\} \tag{8b}$$

$$w_3(r,\theta)=[\frac{\theta^2}{\alpha}-\frac{\alpha}{4}]^2 r\{\frac{\beta r^3}{72+8\phi+\lambda}+K_1r^n+K_2r^{-n}+K_3r^n\ell nr+K_4r^{-n}\ell nr\} \tag{8c}$$

Deflection w_3 represents the borderline case between the real and complex cases. For the isotropic sector it will occur at approximately $\alpha=2.74$ rad. The choice of boundary conditions along the circular edges will determine the constants $K_{1,2,3,4}$ in Eqs. (8), and expressions for bending moments and stresses are obtained from standard equations for polar orthotropic plates.

Results

Isotropic Sector

The deflection expressions given in Eqs. (8) reduce to the case of the isotropic plate when $D_r=D_\theta=D_{r\theta}$ where $D_{r\theta}=D_r\nu_\theta+2D_k$. The deflections and stresses obtained for the isotropic case compare well to available data for various plates, including Woinowsky-Krieger's[7] exact solution for the maximum deflection of a uniformly loaded clamped semicircular isotropic plate. Timoshenko[8] analyses the bending of a uniformly loaded isotropic square plate which has two opposite sides clamped and the other two sides simply supported. The deflection at the plate center, when $\nu=0.3$, is found to be $w=0.00192qs^4/D$ where s is the side of the square. In the present work, an isotropic sector plate clamped on its radial edges and simply supported along its circular edges was chosen such that the length of the center circular line was equal to $(b-a)$, the length of the radial edges. A sector angle $\alpha=\pi/6$ was used, and the deflection obtained with $\nu=0.3$ was $w=0.00192q(b-a)^4/D$, the same result obtained by Timoshenko for the square.

The location of the maximum deflection of the sector should vary with the sector angle. Morley shows this variation for the uniform loaded clamped isotropic case. Figure 2 shows how results obtained using the theorem of minimum potential energy in this study (with $a\ll b$) compared to Morley's data for the pie-shaped sector.

Anisotropic Sector—Rotary Regenerator Matrix

As a particular application of Eqs. (8), the general deflection expressions, a numerical example was chosen. Figure 3 shows some of the results obtained for a gas turbine disk-type rotary regenerator made of a Pyroceram glass ceramic matrix. The structure may be assumed to be cylindrically anisotropic with the following properties:

$$E_r=8\times10^4\text{psi};\ E_\theta=1.3\times10^6\text{psi};$$

$$G_{r\theta}=1.65\times10^5\text{psi},\ \nu_r=0.12,\ \nu_\theta=1.88$$

The configuration taken was: $a=2.0$in.; $b=14.1$ in.; $\alpha=\pi$, and $h=2.8$in.

The load due to pressure drop was $q = 0.5$psi. In addition, a peripheral seal load of 1.0 lb/in. was taken at the outer radius. Pressure and seal forces were assumed to be acting in the same direction in order to obtain conservative results (for worst possible operation conditions). In general, pressure and peripheral seal forces are opposite in direction. Deflections and stresses are shown for the center radial line of the sector. Maximum values along this line are the maximum values for the entire plate.

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Boundary-Layer Flows with Swirl and Large Suction

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Introduction

A NUMBER of devices such as swirl generators, swirl atomizers, rockets, vortex tubes, arc heaters, etc., involve flows with swirl. With these applications in mind, the effects of swirl, zero, or moderate mass transfer on the flow and heat transfer in the low-speed swirling laminar compressible boundary-layer flow over an axisymmetric surface with variable cross section have been investigated by Back[1] and Vimala[2] using the quasilinearization technique. This Note presents an analytical solution for the laminar swirling flow in a tube, a particular type of swirling flow (treated in Refs. 1 and 2) corresponding to a zero longitudinal acceleration parameter (i.e., $\bar{\beta}=0$), with large suction at the surface. Solutions are obtained by making use of the perturbation technique, as has been done by Nanbu.[3] Results compare well with those of Ref. 2, as the value of the suction parameter increases.

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Analysis

The similarity equations governing the low-speed swirling flow of a perfect gas with constant specific heat, viscosity proportional to temperature, Prandtl number unity for the no-slip flow with longitudinal acceleration parameter $\bar{\beta}=0$ and with suction at the surface are[1,2]

$$f''' + ff'' + \bar{\alpha}[G(1-g_w) + g_w - G^2] = 0, \quad G'' + fG' = 0 \tag{1}$$

subject to boundary conditions

$$f(0) = f_w,\ f'(0) = G(0) = 0\ f'(\infty) = G(\infty) = 1 \tag{2}$$

Here f is the dimensionless stream function; G stands for both the normalized swirl velocity and the enthalpy difference ratio; $\bar{\alpha}$ is the swirl parameter, and g_w is the cooling parameter; $f_w = -\rho_w w_w\,(2\Xi)^{1/2}/r\rho_e\mu_e u_e$ is the suction parameter (where $-w_w$ is the suction velocity, and other symbols ρ_w, u_e, μ_e, etc., are defined in Ref. 1); primes denote differentiation with respect to the independent similarity variable Z. Defining Ref. 3:

$$\bar{Z} = f_w Z,\ f(Z) = f_w F(\bar{Z}),\ G(Z) = G(\bar{Z}) \tag{3}$$

and substituting in Eqs. (1) and (2), we have

$$F''' + FF'' + \epsilon_2[G(1-g_w) + g_w - G^2] = 0 \tag{4a}$$

$$G'' + F\,G' = 0 \tag{4b}$$

with

$$F(0) = G(\infty) = 1, \quad F'(0) = G(0) = 0, \quad F'(\infty) = \epsilon_1 \tag{5}$$

where

$$\epsilon_1 = f_w^{-2}, \qquad \epsilon_2 = \bar{\alpha} f_w^{-4} \tag{6}$$

and primes denote differentiation with respect to $\bar{Z}$.

For large suction, f_w will assume large positive values so that ϵ_1 and ϵ_2 are small. In that case, F and G can be expanded in terms of the small perturbation quantities ϵ_1 and ϵ_2 as follows[4]

$$F = F_o + \epsilon_1 F_{11} + \epsilon_2 F_{12} + \epsilon_1^2 F_{21} + \epsilon_2^2 F_{22} + \ldots \tag{7a}$$

$$G = G_o + \epsilon_1 G_{11} + \epsilon_2 G_{12} + \epsilon_1^2 G_{21} + \epsilon_2^2 G_{22} + \ldots \tag{7b}$$

Substitution for F and G, given by Eq. (7), in Eqs. (4) and (5) yields the following sets of differential equations and boundary conditions for F_o, G_o, and F_{ij}, G_{ij} $(i,j=1,2)$:

Zeroth order 0(1)

$$F_o''' + F_o F_o'' = 0; \quad F_o(0) = 1, \quad F_o'(0) = F_o'(\infty) = 0 \tag{8a}$$

$$G_o'' + F_o G_o' = 0; \quad G_o(0) = 0, \quad G_o(\infty) = 1 \tag{8b}$$

First order $0(\epsilon_1)$

$$F_{11}''' + F_o F_{11}'' + F_{11} F_o'' = 0; \quad F_{11}(0) = F_{11}'(0) = 0, \quad F_{11}'(\infty) = 1 \tag{9a}$$

$$G_{11}'' + F_o G_{11}' + F_{11} G_o' = 0; \quad G_{11}(0) = G_{11}(\infty) = 0 \tag{9b}$$

First order $0(\epsilon_2)$

$$F_{12}''' + F_o F_{12}'' + F_{12} F_o'' + G_o(1-g_w) + g_w - G_o^2 = 0; \tag{10a}$$

$$G_{12}'' + F_{12} G_o' + F_o G_{12}' = 0; \tag{10b}$$

$$F_{12}(0) = F'_{12}(0) = F_{12}'(\infty) = G_{12}(0) = G_{12}(\infty) = 0 \tag{10c}$$